$$q_y = -kA_y \frac{\partial T}{\partial y} \tag{4.3}$$

These heat-flow quantities are directed either in the x direction or in the y direction. The total heat flow at any point in the material is the resultant of the qx and qy at that point. Thus the total heat-flow vector is directed so that it is perpendicular to the lines of constant temperature in the material, as shown in Figure 4.1. So if the temperature distribution in the material is known, we may easily establish the heat flow.

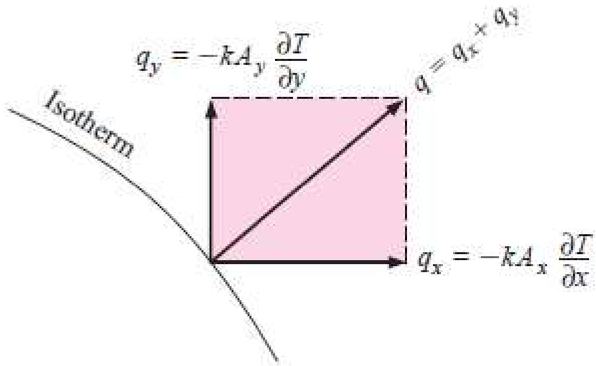


Figure 4.1 Sketch showing the heat flow in two dimensions.

4.1 MATHEMATICAL ANALYSIS OF TWO-DIMENSIONAL HEAT CONDUCTION

We first consider an analytical approach to a two-dimensional problem and then indicate the numerical and graphical methods that may be used to advantage in many other problems.

It is worthwhile to mention here that analytical solutions are not always possible to obtain; indeed, in many instances they are very cumbersome and difficult to use. In these cases numerical techniques are frequently used to advantage.

Consider the rectangular plate shown in Figure 4.2. Three sides of the plate are maintained at the constant temperature T_1, and the upper side has some temperature distribution impressed upon it. This distribution could be simply a constant temperature or something more complex, such as a sine-wave distribution.We shall consider both cases.

4.1.1 Seperation of variable method:

To solve Equation (4.1), the separation-of-variables method is used. The essential point of this method is that the solution to the differential equation is assumed to take a product form

$$T = XY \tag{4.4}$$

Where:

$X = X(x)\ ,\ Y = Y(y)$

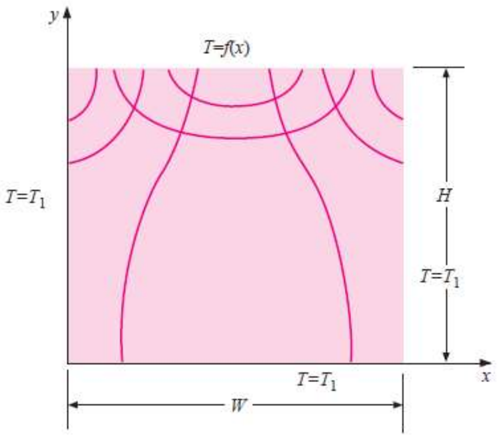


Figure 4.2 Isotherms and heat flow lines in a rectangular plate.

$\frac{\partial T}{\partial x} = \frac{dX}{dx}Y$

$\frac{\partial^2 T}{\partial x^2} = \frac{d^2X}{dx^2}Y$

Similarly,

$\frac{\partial^2 T}{\partial y^2} = \frac{d^2Y}{dy^2}X$

Substitute in equation 4.1

$$\frac{d^2X}{dx^2}Y + \frac{d^2Y}{dy^2}X = 0 \qquad 4.5$$

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \qquad 4.6$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \qquad 4.7$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \qquad 4.8$$

From equation 4.7

$$\frac{d^2X}{dx^2} + X\lambda^2 = 0 \qquad 4.9$$

From equation 4.8

$$\frac{d^2Y}{dy^2} - Y\lambda^2 = 0 \qquad 4.10$$

Solution of equation 4.9

$$X = A\sin\lambda x + B\cos\lambda x \qquad 4.11$$

Solution of equation 4.10

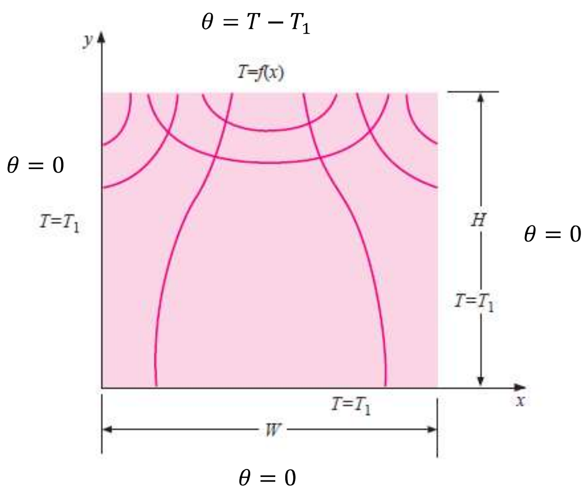
$Y = C \sinh \lambda y + D \cosh \lambda y$ 4.12

General solution of equation 4.1

$T = (A \sin \lambda x + B \cos \lambda x)(C \sinh \lambda y + D \cosh \lambda y)$ 4.13

Let $\theta = T - T_1$

$\theta = (A \sin \lambda x + B \cos \lambda x)(C \sinh \lambda y + D \cosh \lambda y)$



B.C

At $x = 0, \theta = 0$ for all y

$B = 0$

$\theta = (A \sin \lambda x)(C \sinh \lambda y + D \cosh \lambda y)$ 4.14

At $y = 0, \theta = 0$ for all x

$D = 0$

The general solution becomes:

$\theta = (A * C) \sin \lambda x \sinh \lambda y$ 4.15

Let $A * C = E$

At $x = W$, $\theta = 0$ for all y

$\theta = E \sin \lambda x \sinh \lambda y$

$0 = E \sin \lambda W \sinh \lambda y$ 4.16

$\sin \lambda W = 0$

Therefore,

$\lambda = \frac{n\pi}{W}$ where, $n = 0,1,2,3,4, \dots \dots$

$$\theta = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{W} * \sinh\frac{n\pi y}{W} \qquad 4.17$$

To evaluate E_n, sibstitute the last B.C at $y = H$

$$T - T_1 = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{W} * \sinh\frac{n\pi H}{W} \qquad 4.18$$

How to evaluate E_n

This is a Fourier sine series, and the values of the E_n may be determined by expanding the constant temperature difference $T_2 - T_1$ in a Fourier series over the interval $0<x<W$. This series is

$$T_2 - T_1 = T_2 - T_1 \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \sin\frac{n\pi x}{W} \qquad 4.19$$

$$E_n = \frac{2}{\pi}(T_2 - T_1)\frac{1}{\sinh\frac{n\pi H}{W}}\frac{(-1)^{n+1}+1}{n}$$

The final solution:

$$\frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \sin\frac{n\pi x}{W}\frac{\sinh\frac{n\pi y}{W}}{\sinh\frac{n\pi W}{H}} \qquad 4.20$$

4.2 Numerical Method of Analysis

In many practical situations the geometry or boundary conditions are such that an analytical solution has not been obtained at all, or if the solution has been developed, it involves such a complex series solution that numerical evaluation becomes exceedingly difficult. For such situations the most fruitful approach to the problem is one based on finite-difference techniques, the basic principles of which we shall outline in this section.

4.2.1 Finite Difference Tichnique

Consider a two-dimensional body that is to be divided into equal increments in both the *x* and *y* directions, as shown in Figure 4.3. The nodal points are designated as shown, the *m* locations indicating the *x* increment and the *n* locations indicating the *y* increment. We wish to establish the temperatures at any of these nodal points within the body, using Equation (4.1) as a governing condition. Finite differences are used to approximate differential increments in the temperature and space coordinates; and the smaller we choose these finite increments, the more closely the true temperature distribution will be approximated.

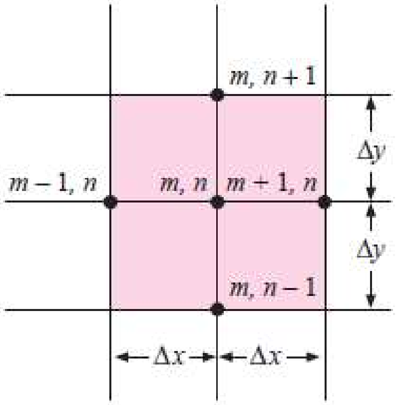


Figure 4.3 Sketch illustrating nomenclature used in two-dimensional numerical analysis of heat conduction.

The temperature gradients may be written as follows:

$$\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} \approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n-1/2} \approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Thus the finite-difference approximation for Equation (4.1) becomes

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$, then

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \qquad 4.21$$

Since we are considering the case of constant thermal conductivity, the heat flows may all be expressed in terms of temperature differentials. Equation (4.21) states very simply that the net heat flow into any node is zero at steady-state conditions. In effect, the numerical finite-difference approach replaces the continuous temperature distribution by fictitious heat-conducting rods connected between small nodal points that do not generate heat. We can also devise a finite-difference scheme to take heat generation into account.We merely add the term $\dot{q}/k$ into the general equation and obtain

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}+\frac{\dot{q}}{k}=0$$

Then for a square grid in which $\Delta x = \Delta y$,

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}+\frac{\dot{q}\Delta x^2}{k}-4T_{m,n}=0 \qquad 4.22$$

To utilize the numerical method, Equation (4.21) must be written for each node within the material and the resultant system of equations solved for the temperatures at the various nodes. A very simple example is shown in Figure 4.4, and the four equations for nodes 1, 2, 3, and 4 would be

$$100+500+T_2+T_3-4T_1=0$$
$$T_1+500+100+T_4-4T_2=0$$

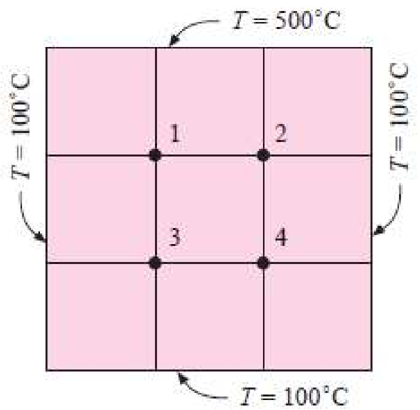


Figure 4.4 Four-node problem.

$$100+T_1+T_4+100-4T_3=0$$
$$T_3+T_2+100+100-4T_4=0$$

These equations have the solution

$$T_1 = T_2 = 250°C \qquad T_3 = T_4 = 150°C$$

Of course, we could recognize from symmetry that $T_1 = T_2$ and $T_3 = T_4$ and would then only need two nodal equations,

$$100 + 500 + T_3 - 3T_1 = 0$$
$$100 + T_1 + 100 - 3T_3 = 0$$

Once the temperatures are determined, the heat flow may be calculated from

$$q = \sum k\,\Delta x\,\frac{\Delta T}{\Delta y}$$

where the ΔT is taken at the boundaries. In the example the heat flow may be calculated at either the 500◦C face or the three 100◦C faces. If a sufficiently fine grid is used, the two values should be very nearly the same. As a matter of general practice, it is usually best to take the arithmetic average of the two values for use in the calculations. In the example, the two calculations yield:

500˚C face:

$$q = -k\frac{\Delta x}{\Delta y}[(250-500)+(250-500)] = 500k$$

100˚C face:

$$q = -k\frac{\Delta y}{\Delta x}[(250-100)+(150-100)+(150-100)+(150-100)+(150-100)+(250-100)] = -500k$$

4.2.2 Energy Balance Method

In many cases, it is desirable to develop the finite-difference equations by an alternative method called the *energy balance method*. As will become evident, this approach enables one to analyze many different phenomena such as problems involving multiple materials, embedded heat sources, or exposed surfaces that do not align with an axis of the coordinate system. In the energy balance method, the finite-difference equation for a node is obtained by applying conservation of energy to a control volume about the nodal region.

Consider a control volume about the interior node m,n of figure 4.5.

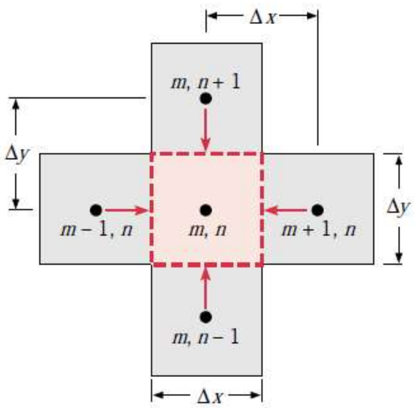


Figure 4.5: conduction to an interior node from its adjoining nodes.

$$-k\Delta y\frac{T_{m,n}-T_{m+1,n}}{\Delta x}-k\Delta y\frac{T_{m,n}-T_{m-1,n}}{\Delta x}-k\Delta x\frac{T_{m,n}-T_{m,n+1}}{\Delta y}-k\Delta x\frac{T_{m,n}-T_{m,n-1}}{\Delta y}=0$$

Taking $\Delta x=\Delta y$, and simplify

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

When the solid is exposed to some convection boundary condition, the temperatures at the surface must be computed differently from the method given above. Consider the boundary shown in Figure 4.5. The energy balance on node *(m, n)* is

$$-k\Delta y\frac{T_{m,n}-T_{m-1,n}}{\Delta x}-k\frac{\Delta x}{2}\frac{T_{m,n}-T_{m,n+1}}{\Delta y}-k\frac{\Delta x}{2}\frac{T_{m,n}-T_{m,n-1}}{\Delta y}=h\Delta y(T_{m,n}-T_{\infty})$$

If $\Delta x=\Delta y$, the boundary temperature is expresses in the equation

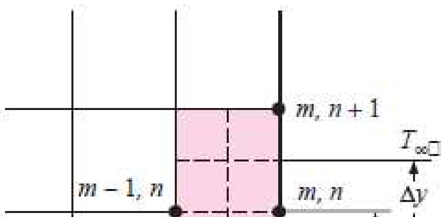
$$T_{m,n}\left(\frac{h\Delta x}{k}+2\right)-\frac{h\Delta x}{k}T_{\infty}-\frac{1}{2}\left(2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}\right)=0 \qquad 4.23$$

An equation of this type must be written for each node along the surface shown in Figure 4.6. So when a convection boundary condition is present, an equation like (4.23) is used at the boundary and an equation like (4.21) is used for the interior points. Equation (4.23) applies to a plane surface exposed to a convection boundary condition. It will not apply for other situations, such as an insulated wall or a corner exposed to a convection boundary condition. Consider the corner section shown in Figure 4.7. The energy balance for the corner section is

$$-k\frac{\Delta y}{2}\frac{T_{m,n}-T_{m-1,n}}{\Delta x}-k\frac{\Delta x}{2}\frac{T_{m,n}-T_{m,n-1}}{\Delta y}=h\frac{\Delta x}{2}\left(T_{m,n}-T_{\infty}\right)+h\frac{\Delta y}{2}\left(T_{m,n}-T_{\infty}\right)$$

If $\Delta x=\Delta y$,

$$2T_{m,n}\left(\frac{h\Delta x}{k}+1\right)-2\frac{h\Delta x}{k}T_{\infty}-\left(T_{m-1,n}+T_{m,n-1}\right)=0 \qquad 4.24$$



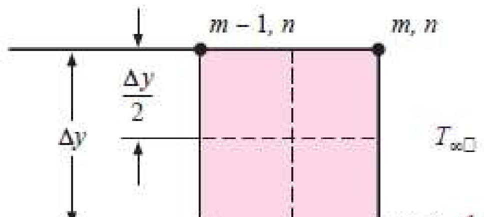


Figure 4.6 Nomenclature for nodal equation with convective boundary condition.

Figure 4.7 Nomenclature for nodal equation with convection at a corner section.

EXAMPLE 4.1: Nine-Node Problem

Consider the square of Figure Example 3-5. The left face is maintained at 100◦C and the top face at 500◦C, while the other two faces are exposed to an environment at 100◦C: $h = 10$ W/m^2 · ◦C and k =10 W/m.◦C .The block is 1 m square. Compute the temperature of the various nodes as indicated in Figure Example 4.1 and the heat flows at the boundaries.

Solution

The nodal equation for nodes 1, 2, 4, and 5 is

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

Node 1:

$$T_2 + T_4 + 500 + 100 - 4T_1 = 0$$

Node 2:

$$T_1 + T_3 + T_5 + 500 - 4T_2 = 0$$

Node 4:

$$T_1 + T_5 + T_7 + 100 - 4T_4 = 0$$

Node 5:

$$T_2 + T_4 + T_6 + T_8 - 4T_5 = 0$$

$$\frac{h\Delta x}{k} = \frac{(10)(1)}{(3)(10)} = \frac{1}{3}$$

Node 3:

$$-k\Delta y\frac{T_3-T_2}{\Delta x}-k\frac{\Delta x}{2}\frac{T_3-500}{\Delta y}-k\frac{\Delta x}{2}\frac{T_3-T_6}{\Delta y}=h\Delta y(T_3-T_\infty)$$

$$T_3\left(\frac{h\Delta x}{k}+2\right)-\frac{h\Delta x}{k}T_\infty-\frac{1}{2}(2T_2+500+T_6)=0$$

$$T_3\left(\frac{1}{3}+2\right)-\frac{1}{3}T_\infty-\frac{1}{2}(2T_2+500+T_6)=0$$

$$2T_2+567+T_6-4.67T_3=0$$

Node 6:

$$-k\Delta y\frac{T_6-T_5}{\Delta x}-k\frac{\Delta x}{2}\frac{T_6-T_3}{\Delta y}-k\frac{\Delta x}{2}\frac{T_6-T_9}{\Delta y}=h\Delta y(T_3-T_\infty)$$

$$T_6\left(\frac{1}{3}+2\right)-\frac{1}{3}T_\infty-\frac{1}{2}(2T_5+T_3+T_9)=0$$

$$2T_5+T_3+T_9+67-4.67T_6=0$$

Node 7:

$$-k\Delta x\frac{T_7-T_4}{\Delta y}-k\frac{\Delta y}{2}\frac{T_7-T_8}{\Delta x}-k\frac{\Delta y}{2}\frac{T_7-100}{\Delta x}=h\Delta x(T_7-T_\infty)$$

$$T_7\left(\frac{1}{3}+2\right)-\frac{1}{3}T_\infty-\frac{1}{2}(2T_4+100+T_8)=0$$

$$2T_4+T_8+167-4.67T_7=0$$

Node 8:

$$-k\Delta x\frac{T_8-T_5}{\Delta y}-k\frac{\Delta y}{2}\frac{T_8-T_7}{\Delta x}-k\frac{\Delta y}{2}\frac{T_8-T_9}{\Delta x}=h\Delta x(T_8-T_\infty)$$

$$T_8\left(\frac{1}{3}+2\right)-\frac{1}{3}T_\infty-\frac{1}{2}(2T_5+T_9+T_7)=0$$

$$2T_5+T_7+T_9+67-4.67T_8=0$$

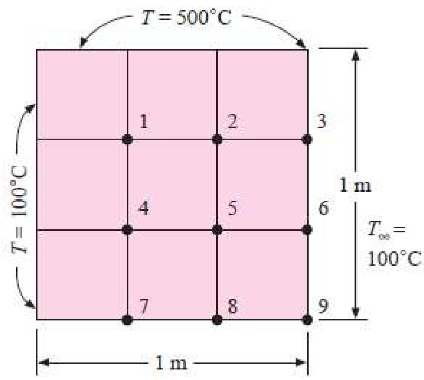


Figure Example 4.1: Nomenclature for Example 4.1.

Node 9:

$$-k\frac{\Delta x}{2}\frac{T_9-T_6}{\Delta y}-k\frac{\Delta y}{2}\frac{T_9-T_8}{\Delta x}=h\frac{\Delta x}{2}(T_9-T_\infty)+h\frac{\Delta y}{2}(T_9-T_\infty)$$

$$2T_9\left(\frac{h\Delta x}{k}+1\right)-2\frac{h\Delta x}{k}T_\infty-(T_6+T_8)=0$$

$$2T_9\left(\frac{1}{3}+1\right)-\frac{2}{3}T_\infty-(T_6+T_8)=0$$

$$T_6+T_8+67-2.67T_9=0$$

We thus have nine equations and nine unknown nodal temperatures. We shall discuss solution techniques shortly, but for now we just list the answers:

Node	**Temperature, ◦C**
1	280.67
2	330.30
3	309.38
4	192.38
5	231.15
6	217.19
7	157.70
8	184.71
9	175.62

The heat flows at the boundaries are computed in two ways: as conduction flows for the 100 and 500◦C faces and as convection flows for the other two faces.

For the 500◦C face,

the heat flow *into* the face is

$$q=\sum k\Delta x\frac{\Delta T}{\Delta y}=(10)\left[(500-280.67)+(500-330.30)+(500-309.38)\left(\frac{1}{2}\right)\right]$$
$q=4843.4$ W/m.

The heat flow *out* of the 100◦C face is

$$q=\sum k\Delta y\frac{\Delta T}{\Delta x}=(10)\left[(280.67-100)+(192.38-100)+(157.7-100)\left(\frac{1}{2}\right)\right]$$
$q=3019$ W/m.

The convection heat flow *out* the right face is given by the convection relation

$$q=\sum h\Delta y(T-T_\infty)$$
$$q=(10)\left(\frac{1}{3}\right)\left[309.38-100+217.19-100+(175.62-100)\left(\frac{1}{2}\right)\right]$$
$q=1214.6$ W/m.

Finally, the convection heat flow *out* the bottom face is

$$q=\sum h\Delta x(T-T_\infty)$$
$$q=(10)\left(\frac{1}{3}\right)\left[(100-100)\left(\frac{1}{2}\right)+(157.7-100)+(184.71-100)+(175.62-100)\left(\frac{1}{2}\right)\right]$$
$q=600.7$ W/m.

The total heat flow out is